

# OFF-DIAGONAL DECAY OF TORIC BERGMAN KERNELS

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**ABSTRACT.** We study the off-diagonal decay of Bergman kernels  $\Pi_{h^k}(z, w)$  and Berezin kernels  $P_{h^k}(z, w)$  for ample invariant line bundles over compact toric projective Kähler manifolds of dimension  $m$ . When the metric is real analytic,  $P_{h^k}(z, w) \simeq k^m \exp -kD(z, w)$  where  $D(z, w)$  is the diastasis. When the metric is only  $C^\infty$  this asymptotic cannot hold for all  $(z, w)$  since the diastasis is not even defined for all  $(z, w)$  close to the diagonal. Our main result is that for general  $C^\infty$  metrics,  $P_{h^k}(z, w) \simeq k^m \exp -kD(z, w)$  as long as  $w$  lies on the  $\mathbb{R}_+^m$ -orbit of  $z$ , and for general  $(z, w)$ ,  $\limsup_{k \rightarrow \infty} \frac{1}{k} \log P_{h^k}(z, w) \leq -D(z^*, w^*)$  where  $D(z, w^*)$  is the diastasis between  $z$  and the translate of  $w$  by  $(S^1)^m$  to the  $\mathbb{R}_+^m$  orbit of  $z$ . These results are complementary to Mike Christ's negative results showing that  $P_{h^k}(z, w)$  does not have off-diagonal decay at "speed"  $k$  if  $(z, w)$  lie on the same  $(S^1)^m$ -orbit.

The problem we are concerned with in this note is to find conditions on a positive Hermitian line bundle  $(L, h) \rightarrow (M, \omega)$  over a Kähler manifold so that the Szegő kernel  $\Pi_{h^k}(z, w)$  for  $H^0(M, L^k)$  has exponential decay at speed  $k$ . We denote the *Berezin kernel* or *normalized Szegő kernel* by

$$P_{h^k}(z, w) := \frac{|\Pi_{h^k}(z, w)|}{\Pi_{h^k}(z, z)^{\frac{1}{2}} \Pi_{h^k}(w, w)^{\frac{1}{2}}} . \quad (1)$$

**Problem** Let  $D_h^*(z, w)$  be the upper semi-continuous regularization of

$$\limsup_{k \rightarrow \infty} \frac{1}{k} (-\log P_{h^k}(z, w)) . \quad (2)$$

Determine  $D_h^*(z, w)$  and in particular determine when it is non-zero.

The minus sign is due to the fact that (1) is pluri-superharmonic in  $z$  and we prefer to deal with pluri-subharmonic functions. It is known that for real analytic metrics,  $P_{h^k}(z, w) \leq C e^{-kD(z, w)}$  for points  $(z, w)$  sufficiently close to the diagonal, where  $D(z, w)$  is the so-called Calabi diastasis (§1.1). Near the diagonal,  $D(z, w) \sim |z - w|^2$ . For general smooth metrics, the sharpest general result is that  $P_{h^k}(z, w) \leq C e^{-A\sqrt{k} \log k}$  for all  $A < \infty$  [Chr13, Chr13B]. This raises the question of whether, for  $C^\infty$  but not real analytic metrics,  $D_h^*(z, w)$  can be strictly negative off the diagonal.

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A stronger condition which arises in several problems (see [RZ]) is whether there exists a pointwise limit

$$\frac{1}{k} \log P_{h^k}(z, w) \rightarrow -D(z, w) \quad (3)$$

for some function  $D(z, w)$  defined near the diagonal in  $M \times \bar{M}$ . If the metric is real analytic, then such a limit does exist and  $D(z, w)$  is the Calabi diastasis of the metric (see §1.1). The diastasis is the real part of the off-diagonal analytic continuation of a local Kaehler potential of  $\omega$  [Ca53]. Existence of a pointwise limit near the diagonal would be surprising if the metric is  $C^\infty$  but not real analytic, since it would define a Calabi diastasis even though the Kaehler potential admits no analytic continuation. One might therefore expect the neighborhood of  $z$  in which the limit (3) exists to be the largest neighborhood of  $z$  in which the Kaehler potential  $\phi$  admits an analytic continuation.

In this note we study these questions in the case of a positive Hermitian holomorphic toric line bundle  $(L, h) \rightarrow (M, \omega_h)$  with  $C^\infty$  metric  $h$ . As recalled in §2, a toric Kähler manifold is a Kähler manifold on which the complex torus  $(\mathbb{C}^*)^m$  acts holomorphically with an open orbit  $M^o$ . We denote by  $\mathbf{T}^m$  the underlying real torus and by  $\mathbb{R}_+^m$  the real subgroup of  $(\mathbb{C}^*)^m$ . We denote a point by  $z = e^{\rho/2 + i\varphi} m_0$  where  $e^{\rho/2}$  denotes the  $\mathbb{R}_+^m$  action and  $e^{i\varphi}$  denotes the  $\mathbf{T}^m$  action. Let  $h = e^{-\phi}$  in a toric holomorphic frame over  $M^o$ . As recalled in §2.3,  $\phi(e^{\rho/2}) = \tilde{\phi}(\rho)$  on the open orbit, where  $\tilde{\phi}$  is convex.

Given two points  $z = e^{\rho_1/2 + i\theta_1}$ ,  $w = e^{\rho_2/2 + i\theta_2}$  we denote by  $z^* = e^{\rho_1/2}$ , resp.  $w^* = e^{\rho_2/2}$  the unique point on the  $\mathbb{R}_+^m$  orbit of  $m_0$  which lie on the same  $\mathbf{T}^m$  orbit as  $z$ , resp.  $w$ . Our main result is that  $D_h^*(z, w) \leq -D(z^*, w^*)$  where  $D(z, w)$  is the Calabi diastasis (see §1.1 and §2.3).

**THEOREM 1.** *Let  $(L, h) \rightarrow (M, \omega)$  be a positive Hermitian toric line bundle over a toric Kaehler manifold. Then if  $z, w \in M^o$ ,*

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \log P_{h^k}(z, w) \leq -D(z^*, w^*) \leq 0,$$

*with  $D(z^*, w^*) = 0$  if and only if  $z^* = w^*$ . Furthermore, if  $z = e^{\rho_1/2 + i\theta}$ ,  $w = e^{\rho_2/2 + i\theta}$  lie on the same  $\mathbb{R}_+^m$  orbit, then one has the pointwise limit (3),*

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log P_{h^k}(z, w) = -D(z, w)$$

*The same asymptotics and upper bounds are valid for all  $z, w \in M$  in the domain of  $D(z, w)$ .*

Thus,  $\limsup_{k \rightarrow \infty} \frac{1}{k} \log P_{h^k}(z, w) < 0$  except on the real codimension  $m$  subset  $\mathcal{M} := \{(z, w) \in M \times M : \exists \vec{\theta} : e^{i\vec{\theta}} w = z\}$  of  $(z, w)$  which lie on the same  $\mathbf{T}^m$  orbit, i.e. have the same  $\rho$  coordinates. We say that  $P_{h^k}(z, w)$  has exponential decay at speed  $k$  except on  $\mathcal{M}$ . In a closely related setting, M. Christ showed that one does not have exponential decay on the analogue of

the set  $\mathcal{M}$ . More precisely, in Theorem 2.1 of [Chr13] he proved that if there exists an open set  $U \subset \mathbb{C}^m$  so that, for any  $\delta > 0$ , there exists a sequence  $k_\nu \rightarrow \infty$  such that  $|B(z, z')| \leq e^{-\epsilon k_\nu}$  for some  $\epsilon > 0$  and for all  $(z, z') \in U$ ,  $|z - z'| \geq \delta$ , then the Kaehler potential  $\phi$  is real analytic on  $U$ . The points where it does not have such decay belong to the analogue of  $\mathcal{M}$ . The result is reviewed in §4 and related to Theorem . Although Christ's result is not stated or proved for toric Kähler manifolds it seems likely that his proof can be modified to apply to them.

Regarding pointwise asymptotics (3), the locus of pairs  $(z, w)$  where this is proved lie on the same  $\mathbb{R}_+^m$  orbit and are also of real codimension  $m$  in  $M \times M$ , and so the set of pairs where we have proved existence of a pointwise limit is very sparse. Although we do not prove it here, it is doubtful that the pointwise limit exists when the angular coordinates of  $z$  and  $w$  are unequal, i.e. if  $z, w \in M^\circ$  but do not lie on the same  $\mathbb{R}_+^m$  orbit. In effect this would require an analytic continuation of  $\phi(z)$  to the full off-diagonal.

Even in simple real analytic cases, the far off-diagonal behavior of the Szegő kernel or Calabi diastasis can be singular. For instance, when  $M = \mathbb{CP}^m$  equipped with its Fubini-Study metric, the Szegő kernel is given in homogeneous coordinates  $Z \in \mathbb{C}^{m+1} \setminus \{0\}$  by

$$\Pi_{h_{FS}^k}(Z, W) = \frac{(Z \cdot \bar{W})^k}{|Z|^k |W|^k}.$$

The Calabi diastasis of the Fubini-Study metric is

$$-D_{FS} = \log \frac{|(Z \cdot \bar{W})|}{|Z||W|} = \frac{1}{k} \log \Pi_{h_{FS}^k}(Z, W).$$

When  $Z \cdot \bar{W} = 0$  the diastasis is infinite and the Szegő kernel vanishes. For  $\mathbb{CP}^1$  in affine coordinates, this occurs when  $w = -\frac{1}{\bar{z}}$ , i.e.  $z = e^{\rho/2 + i\theta}$ ,  $w = e^{-\rho/2 - i\theta}$ . Such points do not lie on the same  $\mathbb{R}_+$  orbit and in higher dimensions anti-podal points  $(Z, W) : Z \cdot \bar{W} = 0$  never lie on the same  $\mathbb{R}_+^m$  orbit.

**0.1. Remarks on the proof.** The proof is based on the explicit formula for the Bergman kernels of a positive toric Hermitian line bundle over a toric Kaehler manifold  $M^m$  of dimension  $m$ . As is well known, there exists a moment map  $\mu : M \rightarrow P$  for the torus action which maps  $M$  to a convex Delzant polytope  $P \subset \mathbb{R}^m$ . The open orbit  $M^\circ$  of the complexified  $(\mathbb{C}^*)^m$ -action maps to the interior of  $P$ , while the inverse image  $\mu^{-1}(\partial P)$  of the boundary of  $P$  is the divisor at infinity  $\mathcal{D} \subset M$ . The toric holomorphic sections  $H^0(M, L^k)$  correspond to monomials  $z^\alpha$  with lattice points  $\alpha \in kP \cap \mathbb{Z}^m$ . In §2.4 and in Lemma 2.3 we review the proof that for  $z, w$  in the open orbit  $M^\circ$ , the toric Bergman kernels for a toric Hermitian metric  $h$  are given by

$$B_{h^k}(z, w) = \sum_{\alpha \in kP} \frac{z^\alpha \bar{w}^\alpha}{Q_{h^k}(\alpha)}, \quad (4)$$

where  $Q_{h^k}(\alpha)$  is the  $L^2$ -norm squared of  $z^\alpha$  with respect to the natural inner product  $\text{Hilb}_k(h)$  defined by  $h$ . As recalled in §2.4,  $B_{h^k}$  is the local expression of  $\Pi_{h^k}$  relative to a local holomorphic frame  $e^k$  of  $L^k$  over  $M^o$ .

The essential point is that the off-diagonal value  $\log P_{h^k}(e^{\rho_1/2}, e^{\rho_2/2})$  equals the diagonal value at  $e^{\frac{1}{2}(\rho_1+\rho_2)}$ . The point  $e^{\frac{1}{2}(\rho_1+\rho_2)}$  is a kind of midpoint with respect to this action of the orbit from  $z$  to  $w$ . The well known diagonal Bergman kernel asymptotics thus determines special cases of the off-diagonal asymptotics. Simple estimates extend the result to  $\mathbf{T}^m$ -orbits of the real slice.

**0.2. Further questions.** A natural question is whether  $D^*(z, w) = D(z^*, w^*)$  for general  $z, w$ . Although the factor  $e^{ik\langle\theta_1-\alpha_2, \alpha\rangle}$  is fast oscillating and can cause significant cancellation, it is not clear whether it can change the exponential decay rate for every sequence of  $k$ 's. Given  $(z, w)$  it is known [SoZ2] that the sum concentrates exponentially fast around one point  $x(z, w) \in P$ . As long as  $\langle\theta_1 - \alpha_2, x(z, w)\rangle$  is irrational, there exists subsequence  $k_n$  so that  $e^{ik_n\langle\theta_1-\alpha_2, x(z, w)\rangle} \rightarrow 1$ . This suggests that  $D^*(z, w) = D(z^*, w^*)$ . However the exponential concentration is not well enough localized to prohibit fast oscillation at nearby lattice points to  $x(z, w)$ , and so the suggestion is not necessarily plausible.

**0.3. Remarks on general metrics.** The original motivation for the off-diagonal decay problem comes from its applications to finite dimensional approximations to solutions of the initial value problem for geodesics in the space of Kaehler metrics in [RZ] and for the boundary value problem in [PhSt, SoZ]. In [RZ] it is conjectured that a solution  $\phi_\tau(z)$  of the initial value problem exists up to time  $T$  if and only if the limit of  $\frac{1}{k}U_{h^k}(i\tau, z, z)$  tends to  $\phi_\tau(z)$  as  $k \rightarrow \infty$ . Without going into the details, the kernel  $U_{h^k}(i\tau, z, z)$  is the value of the Bergman kernel on a certain off-diagonal set. Obviously, the geodesic cannot exist unless the Bergman kernel decays at speed  $k$  on this set and has a limit. If the limit does not exist, one may take the limsup as in (2). Its upper semi-continuous regularization is always well defined and gives a pluri-subharmonic sub-solution of the geodesic equation. Again, this solution is trivial if the speed of decay is smaller than  $k$ . When it equals  $k$  but does not have a pointwise limit, then the regularized limsup will define a singular subsolution, and it is not clear whether it is a weak solution or not. We refer to [RZ] for background and the precise conjecture. For different reasons, the off-diagonal decay rate of Bergman kernels has been studied by M. Christ in several papers [Chr03, Chr13, Chr13B].

it would be interesting to have necessary or sufficient conditions under which the Bergman kernels may satisfy (2) on all but a large codimension set when the metrics are only  $C^\infty$ . We tend to doubt that they exist except in very special (perhaps symmetric) situations such as toric Kähler metrics.

## 1. BACKGROUND ON BERGMAN KERNELS

The Szegő (or Bergman) kernels of a positive Hermitian line bundle  $(L, h) \rightarrow (M, \omega)$  over a Kähler manifold are the kernels of the orthogonal projections  $\Pi_{h^k} : L^2(M, L^k) \rightarrow H^0(M, L^k)$  onto the spaces of holomorphic sections with respect to the inner product  $\text{Hilb}_k(h)$  defined by

$$(s_1, s_2)_{\text{Hilb}_k(h)} = \int_M (s_1(z), s_2(z))_{h^k} \omega_h^m / m!. \quad (5)$$

Thus, we have

$$\Pi_{h^k} s(z) = \int_M \Pi_{h^k}(z, w) \cdot s(w) \frac{\omega_h^m}{m!}, \quad (6)$$

where the  $\cdot$  denotes the  $h^k$ -hermitian inner product at  $w$ . Let  $e_L$  be a local holomorphic frame for  $L \rightarrow M$  over an open set  $U \subset M$  of full measure, and let  $\{s_j^k = f_j e_L^{\otimes k} : j = 1, \dots, d_k\}$  be an orthonormal basis for  $H^0(M, L^k)$  with  $d_k = \dim H^0(M, L^k)$ . Then the Szegő kernel can be written in the form

$$\Pi_{h^k}(z, w) := B_{h^k}(z, w) e_L^{\otimes k}(z) \otimes \overline{e_L^{\otimes k}(w)}, \quad (7)$$

where

$$B_{h^k}(z, w) = \sum_{j=1}^{d_k} f_j(z) \overline{f_j(w)}. \quad (8)$$

The contraction

$$\Pi_{h^k}(z, z) = B_{h^k}(z, z) \|e\|_{h^k}^2 = B_{h^k} e^{-k\phi} \quad (9)$$

balances the exponential growth/decay of the two factors into a power expansion,

$$\Pi_{h^k}(z, z) = B_{h^k}(z, z) e^{-k\phi}(z) = a_0 k^m + a_1(z) k^{m-1} + a_2(z) k^{m-2} + \dots \quad (10)$$

where  $a_0$  is constant; see [Ze1].

**1.1. Kähler potential and Calabi diastasis.** Let  $h = e^{-\phi}$ . When  $\phi$  is real analytic, the analytic continuation  $\phi(z, w)$  of the Kähler potential was used by Calabi [Ca53] in the analytic case to define a Kähler distance function, known as the ‘Calabi diastasis function’

$$D(z, w) := \phi(z, w) + \phi(w, z) - (\phi(z) + \phi(w)). \quad (11)$$

Calabi showed that

$$D(z, w) = d(z, w)^2 + O(d(z, w)^4), \quad dd_w^c D(z, w)|_{z=w} = \omega. \quad (12)$$

When  $\phi$  is only  $C^\infty$  one may try to replace  $\phi(z, w)$  by the *almost analytic extension*  $\phi(z, w)$  of  $\phi$  to  $M \times M$ , defined near the totally real anti-diagonal  $(z, \bar{z}) \in M \times M$  by

$$\phi_{\mathbb{C}}(x + h, x + k) \sim \sum_{\alpha, \beta} \frac{\partial^{\alpha+\beta} \phi}{\partial z^\alpha \partial \bar{z}^\beta}(x) \frac{h^\alpha k^\beta}{\alpha! \beta!}. \quad (13)$$

It is a smooth function defined in a small neighborhood  $(M \times M)_\delta = \{(z, w) : d(z, w) < \delta\}$  of the anti-diagonal in  $M \times M$ , with the right side of (13) as its  $C^\infty$  Taylor expansion along the anti-diagonal, for which  $\bar{\partial}\phi(z, w) = 0$  to infinite order on the anti-diagonal. But the almost analytic extension is not unique and the associated Calabi diastasis can only have a geometric meaning as a germ on the diagonal.

## 2. TORIC BERGMAN KERNELS

We briefly review toric Kähler manifolds and their Bergman kernels; the exposition is similar to that of [SoZ]. A toric Kähler manifold is a Kähler manifold  $(M, J, \omega)$  on which the complex torus  $(\mathbb{C}^*)^m$  acts holomorphically with an open orbit  $M^o$ . Choosing a basepoint  $m_0$  on the open orbit identifies  $M^o \equiv (\mathbb{C}^*)^m$ . We define holomorphic coordinates on the open orbit by giving the point  $z = e^{\rho/2 + i\varphi} m_0$  the coordinates

$$z = e^{\rho/2 + i\varphi} \in (\mathbb{C}^*)^m, \quad \rho, \varphi \in \mathbb{R}^m. \quad (14)$$

The real torus  $\mathbf{T}^m \subset (\mathbb{C}^*)^m$  acts in a Hamiltonian fashion with respect to  $\omega$ . Its moment map  $\mu = \mu_\omega : M \rightarrow P \subset \mathfrak{t}^* \simeq \mathbb{R}^m$  (where  $\mathfrak{t}$  is the Lie algebra of  $\mathbf{T}^m$ ) with respect to  $\omega$  defines a singular torus fibration over a convex lattice polytope  $P$ .

When  $M$  is a smooth toric manifold,  $P$  is a Delzant polytope defined by a set of linear inequalities

$$\ell_r(x) := \langle x, v_r \rangle - \lambda_r \geq 0, \quad r = 1, \dots, d,$$

where  $v_r$  is a primitive element of the lattice and inward-pointing normal to the  $r$ -th  $(m-1)$ -dimensional facet  $F_r = \{\ell_r = 0\}$  of  $P$ . By a facet we mean an  $m-1$ -dimensional face of  $\partial P$ . For  $x \in \partial P$  we denote by

$$\mathcal{F}(x) = \{r : \ell_r(x) = 0\}$$

the set of facets containing  $x$ .

Over the open orbit one has a torus fibration,

$$\mu : M^o \simeq P^o \times \mathbf{T}^m.$$

We let  $x$  denote the Euclidean coordinates on  $P$ . The components  $(I_1, \dots, I_m)$  of the moment map are called action variables for the torus action. The symplectically dual variables on  $\mathbf{T}^m$  are called the angle variables. Given a basis of  $\mathfrak{t}$  or equivalently of the action variables, we denote by  $\{\frac{\partial}{\partial \theta_j}\}$  the corresponding generators (Hamiltonian vector fields) of the  $\mathbf{T}^m$  action. Under the complex structure  $J$ , we also obtain generators  $\frac{\partial}{\partial \rho_j}$  of the  $\mathbb{R}_+^m$  action.

The generators of the  $\mathbf{T}^m$  action vanish on the divisor at infinity,  $\mathcal{D}$ . If  $p \in \mathcal{D}$  and  $\mathbf{T}_p^m$  denotes the isotropy group of  $p$ , then the generating vector fields of  $\mathbf{T}_p^m$  become linearly dependent at  $P$ .

**2.1. Monomial sections.** There exists an orthonormal basis  $\chi_\alpha$  of  $H^0(M, L^k)$  given by eigensections of the  $\mathbf{T}^m$  action for  $\alpha \in k\bar{P}$ . On the open orbit in the coordinates above<sup>1</sup>,

$$\chi_\alpha(z) = z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

Let  $\#P$  denote the number of lattice points  $\alpha \in \mathbb{N}^m \cap P$ . We denote by  $L \rightarrow M$  the invariant line bundle obtained by pulling back  $\mathcal{O}(1) \rightarrow \mathbb{CP}^{\#P-1}$  under the monomial embedding defining  $M$ . A natural basis of the space of holomorphic sections  $H^0(M, L^k)$  associated to the  $k$ th power of  $L \rightarrow M$  is defined by the monomials  $z^\alpha$  where  $\alpha$  is a lattice point in the  $k$ th dilate of the polytope,  $\alpha \in kP \cap \mathbb{N}^m$ . That is, there exists an invariant frame  $e$  over the open orbit so that  $s_\alpha(z) = z^\alpha e^k$ . We denote the dimension of  $H^0(M, L^k)$  by  $N_k$ . We equip  $L$  with a toric Hermitian metric  $h = h_0$  whose curvature  $(1, 1)$  form  $\omega_0 = i\partial\bar{\partial} \log \|e\|_{h_0}^2$  lies in  $\mathcal{H}$ . We often express the norm in terms of a local Kähler potential,  $\|e\|_{h_0}^2 = e^{-\phi}$ , so that  $|s_\alpha(z)|_{h_0^k}^2 = |z^\alpha|^2 e^{-k\phi(z)}$  for  $s_\alpha \in H^0(M, L^k)$ .

The monomials are orthogonal with respect to any such toric inner product and have the norm-squares

$$Q_{h^k}(\alpha) = \|s_\alpha\|_{h^k}^2 = \int_{\mathbb{C}^m} |z^\alpha|^2 e^{-k\phi(z)} dV_\phi(z), \quad (15)$$

where  $dV_\phi = (i\partial\bar{\partial}\phi)^m/m!$ .

**2.2. Coordinates near the divisor at infinity.** To determine the asymptotics of  $P_{h^k}(z, w)$  when at least of  $z, w$  lies on the divisor at infinity, we need expressions for the monomials on the divisor at infinity, i.e. we need to define holomorphic coordinates valid in neighborhoods of points of  $\mathcal{D}$ . We follow [SoZ, STZ]; the following Lemma is proved in [STZ].

**LEMMA 2.1.** *Let  $\alpha \in P \cap \mathbb{Z}^m$ , and  $z \in M_P$ . Then,  $\chi_\alpha(z) = 0$  if and only if*

$$\mu(z) \in \bigcup \{\bar{F}; F \text{ is a facet } \alpha \notin \bar{F}\}. \quad (16)$$

For each vertex  $v_0 \in P$ , we define the chart  $U_{v_0}$  by

$$U_{v_0} := \{z \in M_P; \chi_{v_0}(z) \neq 0\}, \quad (17)$$

Since  $P$  is Delzant, we can choose lattice points  $\alpha^1, \dots, \alpha^m$  in  $P$  such that each  $\alpha^j$  is in an edge incident to the vertex  $v_0$ , and the vectors  $v^j := \alpha^j - v_0$  form a basis of  $\mathbb{Z}^m$ . We define

$$\eta : (\mathbb{C}^*)^m \rightarrow (\mathbb{C}^*)^m, \quad \eta(z) = \eta_j(z) := (z^{v^1}, \dots, z^{v^m}). \quad (18)$$

The map  $\eta$  is a  $\mathbf{T}^m$ -equivariant biholomorphism with inverse

$$z : (\mathbb{C}^*)^m \rightarrow (\mathbb{C}^*)^m, \quad z(\eta) = (\eta^{\Gamma e^1}, \dots, \eta^{\Gamma e^m}), \quad (19)$$

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<sup>1</sup>Throughout the article we use standard multi-index notation, and put  $|\alpha| = \alpha_1 + \dots + \alpha_m$ .

where  $e^j$  is the standard basis for  $\mathbb{C}^m$ , and  $\Gamma$  is an  $m \times m$ -matrix with  $\det \Gamma = \pm 1$  and integer coefficients defined by

$$\Gamma v^j = e^j, \quad v^j = \alpha^j - v_0. \quad (20)$$

Then

$$\chi_{\alpha^j}(z) = \left( \prod \eta_j(z) \right) \chi_{v_0}(z), \quad z \in (\mathbb{C}^*)^m, \quad \left( \prod \eta_j(z) \right) := \prod_{j=1}^m z^{v^j}.$$

The corner of  $P$  at  $v_0$  is transformed to the standard corner of the orthant  $\mathbb{R}_+^m$  by the affine linear transformation

$$\tilde{\Gamma} : \mathbb{R}^m \ni u \rightarrow \Gamma u - \Gamma v_0 \in \mathbb{R}^m, \quad (21)$$

which preserves  $\mathbb{Z}^m$ , carries  $P$  to a polytope  $Q_{v_0} \subset \{x \in \mathbb{R}^m; x_j \geq 0\}$  and carries the facets  $F_j$  incident at  $v_0$  to the coordinate hyperplanes  $= \{x \in Q_{v_0}; x_j = 0\}$ . The map  $\eta$  extends a homeomorphism:

$$\eta : U_{v_0} \rightarrow \mathbb{C}^m, \quad \eta(z_0) = 0, \quad z_0 = \text{the fixed point corresponding to } v_0. \quad (22)$$

By this homeomorphism, the set  $\mu_P^{-1}(\bar{F}_j)$  corresponds to the set  $\{\eta \in \mathbb{C}^m; \eta_j = 0\}$ . If  $\bar{F}$  be a closed face with  $\dim F = m - r$  which contains  $v_0$ , then there are facets  $F_{i_1}, \dots, F_{i_r}$  incident at  $v_0$  such that  $\bar{F} = \bar{F}_{i_1} \cap \dots \cap \bar{F}_{i_r}$ . The subvariety  $\mu_P^{-1}(\bar{F})$  corresponding  $\bar{F}$  is expressed by

$$\mu_P^{-1}(\bar{F}) \cap U_{v_0} = \{\eta \in \mathbb{C}^m; \eta_{i_j} = 0, \quad j = 1, \dots, r\}. \quad (23)$$

When working near a point of  $\mu_P^{-1}(\bar{F})$ , we simplify notation by writing

$$\eta = (\eta', \eta'') \in \mathbb{C}^m = \mathbb{C}^r \times \mathbb{C}^{m-r} \quad (24)$$

where  $\eta' = (\eta_{i_j})$  as in (23) and where  $\eta''$  are the remaining  $\eta_j$ 's, so that  $(0, \eta'')$  is a local coordinate of the submanifold  $\mu_P^{-1}(\bar{F})$ . When the point  $(0, \eta'')$  lies in the open orbit of  $\mu_P^{-1}(\bar{F})$ , we often write  $\eta'' = e^{i\theta'' + \rho''/2}$ .

These coordinates may be described more geometrically as *slice-orbit* coordinates [SoZ]. Let  $P_0 \in \mu_P^{-1}(\bar{F})$  and let  $(\mathbb{C}^*)_{P_0}^m$  denote its stabilizer (isotropy) subgroup. Then there always exists a local slice at  $P_0$ , i.e., a local analytic subspace  $S \subset M$  such that  $P_0 \in S$ ,  $S$  is invariant under  $(\mathbb{C}^*)_{P_0}^m$ , and such that the natural  $(\mathbb{C}^*)^m$  equivariant map of the normal bundle of the orbit  $(\mathbb{C}^*)^m \cdot P_0$ ,

$$[\zeta, P] \in (\mathbb{C}^*)^m \times_{(\mathbb{C}^*)_{P_0}^m} S \rightarrow \zeta \cdot P \in M \quad (25)$$

is a biholomorphism onto  $(\mathbb{C}^*)^m \cdot S$ . The slice  $S$  can be taken to be the image of a ball in the hermitian normal space  $T_{P_0}((\mathbb{C}^*)^m P_0)^\perp$  to the orbit under any local holomorphic embedding  $w : T_{P_0}((\mathbb{C}^*)^m P_0)^\perp \rightarrow M$  with  $w(P_0) = P_0, dw_{P_0} = Id$ . The affine coordinates  $\eta''$  above define the slice  $S = \eta^{-1}\{(z', z''(P_0)) : z' \in (\mathbb{C}^*)^r\}$ . The local ‘orbit-slice’ coordinates are then defined by

$$P = (z', e^{i\theta'' + \rho''/2}) \iff \eta(P) = e^{i\theta'' + \rho''/2}(z', 0) \quad (26)$$



where  $(z', 0) \in S$  is the point on the slice with affine holomorphic coordinates  $z' = (\eta')$ .

For simplicity of notation we suppress the transformation  $\tilde{\Gamma}$  and coordinates  $\eta$ , and we will use the ‘orbit-slice’ coordinates of (26). Thus, we denote the monomials corresponding to lattice points  $\alpha$  near a face  $F$  by

$$\chi_{\alpha', \alpha''}(z', e^{\langle (i\theta'' + \rho'')/2 \rangle}) := (z')^{\alpha'} e^{\langle (i\theta'' + \rho'')/2, \alpha'' \rangle}, \quad (27)$$

where  $\tilde{\Gamma}(\alpha) = (\alpha', \alpha'')$  with  $\alpha''$  in the coordinate hyperplane corresponding under  $\tilde{\Gamma}$  to  $F$  and with  $\alpha'$  in the normal space.

**2.3. Toric Calabi diastasis.** In the toric case, the Kaehler potential  $\phi(z, w)$  is  $\mathbf{T}^m$ -invariant, and so

$$\phi(e^{\rho/2 + i\theta}) = \tilde{\phi}(\rho) \quad (28)$$

where  $\tilde{\phi}(\rho)$  is a smooth convex function on  $\mathbb{R}^m$ . We also write  $\phi(z) = F(|z|^2)$  where  $F(e^\rho) = \tilde{\phi}(\rho)$ . Then

$$F_{\mathbb{C}}(z \cdot \bar{w}) = \text{the almost analytic extension of } F(|z|^2) \text{ to } M \times M. \quad (29)$$

Hence,

$$-D(z, w) = F(z \cdot \bar{w}) + F(w \cdot \bar{z}) - F(|z|^2) - F(|w|^2). \quad (30)$$

As an example, the Bargmann-Fock(-Heisenberg) Kaehler potential is  $|z|^2$  and the analytic extension is  $F_{BF, \mathbb{C}}(z, w) = z \cdot \bar{w}$  and  $D(z, w) = |z|^2 + |w|^2 - 2\operatorname{Re} z \bar{w} = |z - w|^2$ . The example of the Fubini-Study metric on  $\mathbb{CP}^m$  was discussed in the introduction.

We now give formulae for the diastasis when two points lie on the same  $\mathbb{R}_+^m$  orbit. If we write  $z = e^{\rho_1/2 + i\theta_1}$ ,  $w = e^{\rho_2/2 - i\theta_2}$ , and  $\tilde{F}(\rho) = F(e^\rho)$ , then

**LEMMA 2.2.** • *When  $z, w \in M^\circ$  lie on the same  $\mathbb{R}_+^m$  orbit, i.e. if  $\theta_1 = \theta_2$  then*

$$-\frac{1}{2}D(z, w) = \tilde{F}\left(\frac{1}{2}(\rho_1 + \rho_2)\right) - \frac{1}{2}(\tilde{F}(\rho_1) + \tilde{F}(\rho_2)). \quad (31)$$

• *If  $z, w \in F$  for some face  $F$  of  $P$ , then in the coordinates (26) with  $z = (0, e^{i\theta_1'' + \rho_1'/2})$ ,  $w = (0, e^{i\theta_2'' + \rho_2'/2})$ ,*

$$-\frac{1}{2}D(z, w) = \tilde{F}\left(\frac{1}{2}(\rho_1'' + \rho_2'')\right) - \frac{1}{2}(\tilde{F}(\rho_1'') + \tilde{F}(\rho_2'')). \quad (32)$$

Since  $F$  is convex, the right sides are negative, i.e.  $D(z, w) > 0$ . The formula for  $z, w \in F$  reflects the fact that the diastasis of a complex submanifold is the restriction of the diastasis on the ambient manifold [Ca53].

**2.4. Toric Bergman kernels.** We now justify (4).

**LEMMA 2.3.** *The toric Bergman kernel is given on the open orbit by*

$$B_{h^k}(z, w) = \sum_{\alpha \in kP} \frac{z^\alpha \bar{w}^\alpha}{Q_{h^k}(\alpha)}. \quad (33)$$

*Proof.* We recall that  $\chi_\alpha(z) = z^\alpha$  is the local representative of  $s_\alpha$  in the open orbit with respect to an invariant frame. Since  $\{\frac{\chi_\alpha}{\sqrt{Q_{h^k}(\alpha)}}\}$  is the local expression of an orthonormal basis, we have

$$B_{h^k}(z, w) = \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{\chi_\alpha(z) \overline{\chi_\alpha(w)}}{Q_{h^k}(\alpha)}$$

hence

$$\hat{\Pi}_{h^k}(z, 0; w, 0) = \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{\chi_\alpha(z) \overline{\chi_\alpha(w)} e^{-k(\phi(z) + \phi(w))/2}}{Q_{h^k}(\alpha)}. \quad (34)$$

In the case of a toric variety with  $0 \in \bar{P}$ , there exists a frame  $e$  such that  $s_\alpha(z) = z^\alpha e$  on the open orbit, and then the Bergman kernel takes the form (4).  $\square$

### 3. OFF-DIAGONAL DECAY: PROOF OF THEOREM 1

We break up the discussion into three cases accordingly as  $z, w \in M^o$ , or one (or both) lie on  $\mathcal{D}$ . The discussion is similar to that of the diastasis in §2.3.

**3.1. Case (i):**  $z, w \in M^o$ . We now estimate  $|B_{h^k}(z, w)|$  on a toric Kähler manifold when  $z, w \in M^o$ . We write  $z = e^{\rho/2 + i\theta}$  where  $\rho \in \mathbb{R}^m$  and  $e^{i\theta} \in \mathbf{T}^m$  and (as in (28))  $\tilde{\phi}(\rho) = \phi(e^{\rho/2})$ . Then  $z^\alpha = e^{\langle \alpha, \rho_1 \rangle / 2 + i\langle \theta_1, \alpha \rangle}$  and  $w^\alpha = e^{\langle \alpha, \rho_2 \rangle / 2 + i\langle \theta_2, \alpha \rangle}$ . Hence

$$|B_{h^k}(z, w)| = \left| \sum_{\alpha \in kP} \frac{e^{\langle \alpha, \frac{1}{2}(\rho_1 + \rho_2) \rangle + i\langle \alpha, \theta_1 - \theta_2 \rangle}}{Q_{h^k}(\alpha)} \right|. \quad (35)$$

Thus,

$$P_{h^k}(z, w) = \left| \sum_{\alpha \in kP} \frac{e^{\langle \alpha, \frac{1}{2}(\rho_1 + \rho_2) \rangle + i\langle \alpha, \theta_1 - \theta_2 \rangle}}{Q_{h^k}(\alpha)} \right| e^{-\frac{1}{2}k(\tilde{\phi}(\rho_1) + \tilde{\phi}(\rho_2))}. \quad (36)$$

We first consider a special case:

**3.2.  $z$  and  $w$  have the same  $\theta$  coordinate.** Suppose that  $z = e^{\rho_1/2 + i\theta}$  and  $w = e^{\rho_2/2 + i\theta}$ .

$$|B_{h^k}(z, w)| = \sum_{\alpha \in kP} \frac{e^{\langle \alpha, \frac{1}{2}(\rho_1 + \rho_2) \rangle}}{Q_{h^k}(\alpha)}. \quad (37)$$

This is equivalent to an on-diagonal value of the Bergman kernel:

$$B_{h^k}(e^{\rho'/2}, e^{\rho'/2}) = \sum_{\alpha \in kP} \frac{e^{\langle \alpha, \rho' \rangle}}{Q_{h^k}(\alpha)},$$

with  $\rho' = \frac{1}{2}(\rho_1 + \rho_2)$ . By the diagonal Bergman kernel asymptotics (10),

$$|P_{h^k}(e^{\rho_1/2+i\theta}, e^{\rho_2/2+i\theta})| \simeq k^m e^{k\tilde{\phi}(\frac{1}{2}(\rho_1+\rho_2))} e^{-k\frac{1}{2}(\tilde{\phi}(\rho_1)+\tilde{\phi}(\rho_2))}. \quad (38)$$

Since  $\tilde{\phi}$  is convex,

$$\tilde{\phi}\left(\frac{1}{2}(\rho_1 + \rho_2)\right) < \frac{1}{2}(\tilde{\phi}(\rho_1) + \tilde{\phi}(\rho_2)).$$

It follows that

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log |P_{h^k}(e^{\rho_1/2+i\theta}, e^{\rho_2/2+i\theta})| = \tilde{\phi}\left(\frac{1}{2}(\rho_1 + \rho_2)\right) - \frac{1}{2}(\tilde{\phi}(\rho_1) + \tilde{\phi}(\rho_2)) < 0.$$

*Remark:* As mentioned in the introduction, the assumption that  $z, w$  have the same  $\theta$ -coordinate could be put more intrinsically by saying that  $z, w$  lie on the same orbit of the  $\mathbb{R}_+$ -action defined by the  $(\mathbb{C}^*)^m$  action.

**3.3.  $z, w \in M^o$  but do not lie on the same  $\mathbb{R}_+^k$  orbit.** It is obvious that

$$\begin{aligned} |B_{h^k}(z, w)| &= \left| \sum_{\alpha \in kP} \frac{e^{\langle \alpha, \frac{1}{2}(\rho_1+\rho_2) \rangle + i\langle \alpha, \theta_1 - \theta_2 \rangle}}{Q_{h^k}(\alpha)} \right| \\ &\leq \sum_{\alpha \in kP} \frac{e^{i\langle \alpha, \frac{1}{2}(\rho_1+\rho_2) \rangle}}{Q_{h^k}(\alpha)}. \end{aligned} \quad (39)$$

Arguing as in the previous case,

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \log |P_{h^k}(e^{\rho_1/2+i\theta}, e^{\rho_2/2+i\theta})| \leq \tilde{\phi}\left(\frac{1}{2}(\rho_1 + \rho_2)\right) - \frac{1}{2}(\tilde{\phi}(\rho_1) + \tilde{\phi}(\rho_2)) < 0.$$

**3.4. Case (ii)  $z \in M^o, w \in \mathcal{D}$ .** In this, case, we must use the slice-orbit coordinates adapted to the face  $F$  containing  $w$ . Of course,  $z, w$  lie on different  $\mathbb{R}^m$  orbits (even  $(\mathbb{C}^*)^m$ -orbits), and we can only give an upper bound then. We use the coordinates in (26), so that  $z = (z', e^{i\theta'_1 + \rho'_1/2})$  and  $w = (0, e^{i\theta''_2 + \rho''_2/2})$  and further write  $z' = e^{\rho'/2 + i\theta'}$ . Then,

$$B_{h^k}(z, w) = \sum_{\alpha \in kP} \frac{e^{i\langle \alpha', \rho' + i\theta' \rangle} e^{\langle \alpha'', (\rho''_1/2 + \rho''_2/2) + i\langle \alpha'', (\theta''_1 - \theta''_2) \rangle}}{Q_{h^k}(\alpha)}. \quad (40)$$

If  $\theta' = 0, \theta''_1 = \theta''_2$  then

$$|B_{h^k}(z, w)| = \sum_{\alpha \in kP} \frac{e^{i\langle \alpha', \rho' \rangle} e^{\langle \alpha'', (\rho''_1/2 + \rho''_2/2) \rangle}}{Q_{h^k}(\alpha)}. \quad (41)$$

Otherwise this is an upper bound. This is the value on the diagonal of the Bergman kernel at the point with coordinates

$$(e^{\frac{1}{2}\rho'}, e^{\frac{1}{4}(\rho''_1 + \rho''_2)}).$$

We again use the diagonal Bergman kernel asymptotics (10). We cannot use the open-orbit Kähler potential since it is not well-defined on the divisor  $\mathcal{D}$ . But the diastasis is independent of the choice of Kähler potential and by the previous calculation we find that  $\frac{1}{k} \log P_{h^k}(z, w)$  tends to  $D(z, w)$ .

#### 4. OFF-DIAGONAL BERGMAN KERNEL FOR M. CHRIST'S METRICS

In this section, we briefly explain the modification of Theorem which applies to M. Christ's metrics [Chr13]. He considers Kaehler metrics on  $\mathbb{C}^m$  whose potentials  $\phi(z) = \phi(x + iy)$  are functions of  $x$  alone. Such metrics are analogous to toric metrics with the torus replaced by  $\mathbb{R}^m$ . More precisely, we think of  $\mathbb{R}_x^m$  and  $\mathbb{R}_y^m$  as two additive subgroups of  $\mathbb{C}^m$ , both with non-compact orbits. The difference between toric Kaehler metrics and those of [Chr13] is that the  $e^{i\theta}$  action with compact torus orbits is replaced by the additive  $i\mathbb{R}^n$  action with non-compact orbits, and the metric is invariant under this action. Since these metrics are similar to toric metrics except that the torus is 'unravelling' to  $\mathbb{R}^m$ , we refer to them as 'non-compact toric' metrics. To our knowledge, there does not exist a standard term in complex geometry.

For non-compact toric Kaehler manifolds, one has an  $\mathbb{R}^m$  action replacing the  $\mathbb{T}^m$  actions and therefore a continuous spectrum of joint eigenfunctions. The image of the moment map of the  $\mathbb{R}^m$  action is all of  $\mathbb{R}^m$  rather than a convex Delzant polytope  $P$ . The exponents  $\alpha$  are lattice points in  $kP$  in the toric case, but are simply elements  $\alpha \in \mathbb{R}^m$  for every power of the line bundle in the non-compact case. In the toric case one writes the monomial  $e^{\langle \alpha, \rho/2 + i\theta \rangle}$  in angle-action coordinates and in the non-compact case one writes the joint eigenfunctions as  $e^{\langle \alpha, \rho/2 + iy \rangle}$  with  $\alpha, y \in \mathbb{R}^m$ .

The weighted Hilbert space in the non-compact case for the  $\lambda$ -th power of the positive Hermitian line bundle  $L \rightarrow \mathbb{C}^m$  is the set of entire holomorphic functions  $f$  so that

$$\|f\|_{L^2(X, L^\lambda)}^2 = \int_{\mathbb{C}^m} |f(z)|^2 e^{-\lambda\phi(z)} dm(z) < \infty. \quad (42)$$

It is assumed that the curvature form of  $\phi$  is strictly positive and uniformly bounded above and below. Thus the real Hessian of  $\phi$  is a positive matrix  $\text{Hess}(\phi)(x)$  satisfying

$$C^{-1}|v|^2 \leq \langle \text{Hess}(\phi)(x)v, v \rangle \leq C|v|^2.$$

Note that such a Kaehler metric is quite different from the usual Bargmann-Fock type Kaehler metrics such as  $|z|^2$ , since the latter are invariant under the compact torus action (i.e. are non-compact toric Kaehler manifolds in the standard sense).

Theorem 2.1 of [Chr13] says that if there exists an open set  $U \subset \mathbb{C}^m$  so that, for any  $\delta > 0$ , there exists a sequence  $\lambda_\nu \rightarrow \infty$  such that  $|B(z, z')| \leq e^{-\epsilon\lambda_\nu}$  for some  $\epsilon > 0$  and for  $(z, z') \in U$ ,  $|z - z'| \geq \delta$ , then the Kaehler potential  $\phi$  is real analytic on  $U$ .

Of course, the joint eigenfunctions  $e^{\langle \alpha, x/2 + iy \rangle}$  do not lie in  $L^2$  since the integrals

$$\int_{\mathbb{R}^{2m}} e^{-\langle \alpha, x \rangle} e^{-\lambda \phi(x)} dx dy$$

diverges due to the lack of damping in the  $y$  variable. We may however express the Bergman kernel in terms of a generalized orthonormal basis of joint eigenfunctions since

$$\int_{\mathbb{R}^{2m}} e^{\langle (\alpha+\beta), x/2 \rangle} e^{i\langle \alpha-\beta, y \rangle} e^{-\lambda \phi(x)} dx dy = \delta(\alpha - \beta) \left( \int_{\mathbb{R}^m} e^{\langle \alpha, x \rangle} e^{-\lambda \phi(x)} dx \right).$$

Analogously to (15), we write

$$Q_{\lambda\phi}(\alpha) = \int_{\mathbb{R}^m} e^{\langle \alpha, x \rangle} e^{-\lambda \phi(x)} dx. \quad (43)$$

**PROPOSITION 4.1.** *The Bergman kernel  $B_\lambda(z, w)$  of Christ's metric has the form,*

$$B_\lambda(z, w) = \int_{\mathbb{R}^m} e^{\langle \alpha, (x+x')/2 + i(y-y') \rangle} \frac{d\alpha}{Q_{\lambda\phi}(\alpha)}.$$

In the present terminology, the Bergman kernel is the kernel which is holomorphic in  $z$ , anti-holomorphic in  $w$  and which represents the Szegő kernel with respect to a local frame  $e_L^\lambda$ . The setting of the trivial line bundle  $L = \mathbb{C}^m \times \mathbb{C} \rightarrow \mathbb{C}^m$  means that the constant function 1 can be used as a frame, but its norm is  $e^{-\lambda \phi(z)/2}$ . Thus the Szegő kernel is

$$\Pi_\lambda(z, w) = B_\lambda(z, w) e^{-\lambda \phi(z)/2} e^{-\lambda \phi(w)/2}. \quad (44)$$

Now let us suppose that  $y = y'$  but  $x \neq x'$ . Then

$$B_\lambda(z, w) = \int_{\mathbb{R}^m} e^{\langle \alpha, (x+x')/2 \rangle} \frac{d\alpha}{Q_{\lambda\phi}(\alpha)}.$$

As in the toric case, we recognize that this is closely related to the diagonal value of the Bergman kernel at the point  $(\frac{x-x'}{2}, y)$  for any  $y$ , i.e.

$$B_\lambda\left(\left(\frac{x+x'}{2}, y\right), \left(\frac{x+x'}{2}, y\right)\right) = \int_{\mathbb{R}^m} e^{\lambda \langle \alpha, (x+x')/2 \rangle} \frac{d\alpha}{Q_{\lambda\phi}(\alpha)}.$$

However as above, the off-diagonal Szegő kernel does not equal the corresponding on-diagonal Szegő kernel at the mid-point but differs by the metric

factor. That is,

$$\begin{aligned}
\Pi_\lambda((x/2 + iy, x'/2 + iy') &= B_\lambda(x/2 + iy, x'/2 + iy') e^{-\lambda\phi(z)/2} e^{-\lambda\phi(z')/2} \\
&= B_\lambda((\frac{x+x'}{2}, y), (\frac{x+x'}{2}, y)) e^{-\lambda\phi(\frac{x+x'}{2}, y)} \\
&\quad \times \left( e^{-\lambda\phi(z)/2} e^{-\lambda\phi(z')/2} e^{\lambda\phi(\frac{x+x'}{2}, y)} \right) \\
&= \Pi_\lambda((\frac{x+x'}{2}, y), (\frac{x+x'}{2}, y)) \left( e^{-\lambda\phi(z)/2} e^{-\lambda\phi(z')/2} e^{\lambda\phi(\frac{x+x'}{2}, y)} \right) \\
&= \Pi_\lambda((\frac{x+x'}{2}, y), (\frac{x+x'}{2}, y)) \left( e^{-\lambda\phi(x)/2} e^{-\lambda\phi(x')/2} e^{\lambda\phi(\frac{x+x'}{2})} \right).
\end{aligned} \tag{45}$$

By the Bergman kernel asymptotics,

$$\Pi_\lambda((\frac{x+x'}{2}, y), (\frac{x+x'}{2}, y)) \sim \lambda^m + O(\lambda^{m-1}).$$

On the other hand, by convexity of  $\phi(x)$ ,

$$e^{-\lambda\phi(x)/2} e^{-\lambda\phi(x')/2} e^{\lambda\phi(\frac{x+x'}{2})}$$

is exponentially decaying at speed  $\lambda$ , since

$$\phi(x)/2 + \phi(x')/2 \geq \phi(\frac{x+x'}{2}).$$

This argument does not give exponential decay when  $x = x'$  and therefore is consistent with the results of [Chr13].

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